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Information theory in the study of anisotropic radiation

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Abstract. Information theory is used to perform a thermodynamic study of nonequilibrium anisotropic radiation. We limit our analysis to a second-order truncation of the moments, obtaining a distribution function which leads to a natural closure of the hierarchy of radiative transfer equations in the so-called variable Eddington factor scheme. Some Eddington factors appearing in the literature can be recovered as particular cases of our two-parameter Eddington factor. We focus our attention on the study of the thermodynamic properties of such systems and relate it to recent nonequilibrium thermodynamic theories. Finally, we comment on the possibility of introducing a nonequilibrium chemical potential for photons.

1. Introduction

The study of radiation hydrodynamics [1] has proven to be of great interest in astrophysics, cosmology and plasma physics. The radiative transfer equation for the specific radiation intensity $I(\boldsymbol{r}, t, \nu, \boldsymbol{\Omega}) = h\nu c\, n(\boldsymbol{r}, t, \nu, \boldsymbol{\Omega})$, where $n(\boldsymbol{r}, t, \nu, \boldsymbol{\Omega})$ is the occupation number of photons with frequency ν moving in direction $\boldsymbol{\Omega}$, is in many practical situations too involved to be solved analytically. What is usually done is to consider the equations for the moments of $I(\boldsymbol{r}, t, \nu, \boldsymbol{\Omega})$ up to a given order m [2, 3]. However, due to the dependence of the equation for the moment m on the moment $m+1$, one needs to introduce a closure relation. If only the energy density e ($m = 0$) and energy flux $\boldsymbol{J}_E$ ($m = 1$) are considered, one must introduce a closure relation for the pressure tensor $\boldsymbol{P}_E$ ($m = 2$). Hence, in this approximation the relevant physical quantities are the angular moments of the intensity (note that in the following we consider as variables the moment of the photons $\boldsymbol{p} = \boldsymbol{\Omega}\, h\nu/c = p\boldsymbol{c}/c$ instead of the frequency ν and the solid angle $\boldsymbol{\Omega}$)

$$e(\boldsymbol{r}, t) = \frac{1}{h^3}\int pc\, n(\boldsymbol{r}, \boldsymbol{p}, t)\, \mathrm{d}^3\boldsymbol{p} \tag{1}$$

$$\boldsymbol{J}_E(\boldsymbol{r}, t) = \frac{1}{h^3}\int pc\boldsymbol{c}n(\boldsymbol{r}, \boldsymbol{p}, t)\, \mathrm{d}^3\boldsymbol{p} \tag{2}$$

$$\boldsymbol{P}_E(\boldsymbol{r}, t) = \frac{1}{h^3}\int \boldsymbol{p}\boldsymbol{c}n(\boldsymbol{r}, \boldsymbol{p}, t)\, \mathrm{d}^3\boldsymbol{p} \tag{3}$$

namely the energy density, energy flux and pressure tensor. It is also convenient to define the following normalized quantities

$$\boldsymbol{f}_E = \frac{\boldsymbol{J}_E}{ec} \qquad \boldsymbol{T}_E = \frac{\boldsymbol{P}_E}{e}. \tag{4}$$

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The closure relation is performed by the introduction of the so-called Eddington factor $\chi(f_E)$, defined as the eigenvalue of the pressure tensor corresponding to the eigenvector $\boldsymbol{n}$ (unitary vector in the direction of the energy flux), i.e.

$$\boldsymbol{T}_E\boldsymbol{n} = \chi\boldsymbol{n}. \tag{5}$$

This definition leads to the relation

$$\boldsymbol{T}_E = \frac{1-\chi}{2}\mathsf{I} + \frac{3\chi-1}{2}\boldsymbol{nn} \tag{6}$$

where I is the identity matrix. In the limit of isotropic radiation (Eddington limit) $\chi(0) = \frac{1}{3}$, while in the free streaming case $\chi(1) = 1$.

In addition, one can define in an analogous way the angular moments of the occupation number $n(\boldsymbol{r}, \boldsymbol{p}, t)$, namely the total photon number density, the photon flow and the flux of the particle flow

$$\rho(\boldsymbol{r}, t) = \frac{1}{h^3}\int_0^\infty n(\boldsymbol{r}, \boldsymbol{p}, t)\,\mathrm{d}^3\boldsymbol{p} \tag{7}$$

$$\boldsymbol{J}_N(\boldsymbol{r}, t) = \frac{1}{h^3}\int_0^\infty \boldsymbol{c}n(\boldsymbol{r}, \boldsymbol{p}, t)\,\mathrm{d}^3\boldsymbol{p} \tag{8}$$

$$\boldsymbol{P}_N(\boldsymbol{r}, t) = \frac{1}{h^3}\int_0^\infty \boldsymbol{cc}n(\boldsymbol{r}, \boldsymbol{p}, t)\,\mathrm{d}^3\boldsymbol{p} \tag{9}$$

and the corresponding normalized quantities

$$\boldsymbol{f}_N = \frac{\boldsymbol{J}_N}{\rho c} \qquad \boldsymbol{T}_N = \frac{\boldsymbol{P}_N}{\rho c^2}. \tag{10}$$

Whenever the angular and frequency dependence of the radiation intensity (or the occupation number) factorize, the two sets of moments are not independent, but verify the relations

$$\boldsymbol{f}_E = \boldsymbol{f}_N \qquad \boldsymbol{T}_E = \boldsymbol{T}_N. \tag{11}$$

This fact is implicitly assumed in most papers on the subject [4]. In many instances, however, the frequency is not integrated [2, 5] and, thus, in this situation the two sets of moments also coincide. However, all the previous quantities would be frequency dependent, except, again, in the case where radiation intensity factorizes into a frequency-dependent part and angular-dependent part.

There is a great amount of different Eddington factors in the literature introduced following physically different approaches (see [2] for a review). Among the different approaches to obtain variable Eddington factors, some authors have used a maximum entropy principle, both from a macroscopic [6, 7] and microscopic point of view [4, 5, 8]. From a macroscopic viewpoint, balance equations for the energy density and energy flux have been considered, and an entropy principle has been introduced to exploit these constraints, leading to the so-called Lorentz's Eddington factor that Levermore [2] obtained. In [8] the same result was obtained from the point of view of information theory with fixed energy flux. However, as pointed out in [9], this anisotropic Eddington factor corresponds to a Lorentz transformation of the equilibrium one so that the anisotropy is only due to the fact that the system is being observed from a moving reference frame. Thus, this Eddington factor does not describe a real out-of-equilibrium situation.

On the other hand, both Minerbo [5] and Fu [4] obtained different Eddington factors, considering as a constraint the photon flux instead of the energy flux. In this case, a reference frame where the anisotropy disappears cannot exist.

In this paper, we apply information theory to generalize the two situations mentioned above, by simultaneously considering the energy flux and photon flow as independent variables. Hence, the previous results can be recovered as limiting cases, while some new situations can be analysed. In addition, information theory allows a complete thermodynamic study of radiation out of equilibrium. The closure scheme previously described, in which the pressure tensor is written as a function of the energy flux, departs from the hypothesis of local equilibrium, which implies that the distribution in momentum space is locally (i.e. for each position) the same as an equilibrium distribution. However, radiation usually has a distribution markedly different from a black-body distribution, and this hypothesis must, therefore, be abandoned in radiative transfer problems [10]. The structure of classical nonequilibrium thermodynamics (for example, bilinear forms for the entropy production rate) also presents a lack of consistency for radiation [10]. When the radiation field is strongly anisotropic, the mean-free path of the photons is large and the set of macroscopic quantities describing the local state of the photon gas arises from interactions occurring over large regions, whereas when the photon gas is in equilibrium the interactions which thermalize photons and matter take place in a specific volume. Thus, to describe the radiation gas, the appropriate set of quantities must contain information about its angular distribution [4].

This can be done in the closure scheme described above by including the fluxes among the set of thermodynamic variables with the help of information theory. This formalism can be used as a heuristic method to find a distribution function consistent with the information available about the system. We will also be able to analyse the influence of the dissipative fluxes in the nonequilibrium equations of state. This procedure to study nonequilibrium equations of state has already been used in [8, 11, 12] in the case of an ideal gas and in [13] to study heat conduction in a boson gas. Indeed, from the point of view of nonequilibrium thermodynamics, the meaning of the fundamental thermodynamic quantities in nonequilibrium states is a basic challenge, so it deserves attention from all possible points of view.

The plan of the paper is as follows. In section 2, we apply the information theoretical formalism to the study of radiation (within the low-occupation number approximation) under an energy and photon flux, considered as independent variables. Hence, we can obtain a two-parameter Eddington factor depending on both fluxes and generalized equations of state. We can also recover Lorentz's and Minerbo's Eddington factors in the proper limits. In section 3 we analyse the possibility of introducing a nonequilibrium chemical potential for photons and its physical consequences. The last section is devoted to reviewing the main conclusions of the paper.

2. Anisotropic radiation under energy and particle fluxes

Information theory was introduced in 1957 by Jaynes [14, 15] in statistical mechanics in order to provide a probabilistic basis to equilibrium thermodynamics. However, although the foundations of the use of the informational entropy functional for nonequilibrium situations are far from being trivial, it has also been applied to nonequilibrium situations (see [16] and references therein). The method asserts that the steady state of a system, defined by the values of a set of macroscopic constraints, is the most (microscopically) disordered state compatible with these constraints, while disorder is measured by means of Shannon's entropy, which is defined as follows. Let $\mathcal{N}$ be the number of microstates compatible with the macroscopic constraints acting on the system and let p_i be the probability of a given microstate i. The informational entropy is given by:

$$S = -k_B \sum_{i=1}^{\mathcal{N}} p_i \ln p_i \tag{12}$$

where k_B is the Boltzmann's constant and providing that the normalization condition $\sum_i p_i = 1$ is fulfilled. By maximizing (12) subject to the constraints (mean values of extensive quantities controlled in a given experiment) the probability of each microstate is obtained. This method realizes a probability assignment which is as unbiased as possible and avoids any unwarranted assumption beyond the information contained in the constraints. The probabilities p_i corresponding to equilibrium ensembles can be easily derived from this postulate of maximum entropy as shown in [17].

In nonequilibrium situations, one considers that the generalized entropy functional (12) depends both on the equilibrium and nonequilibrium constraints acting on the system and the corresponding probabilities p_i are obtained by the maximization of this entropy. These nonequilibrium constraints may be, for instance, the heat flux or the viscous traceless pressure tensor.

If quantum systems are considered, the statistical entropy can be calculated in terms of the occupation number n_i according to [18]

$$S = k_B \sum_{i=1}^{\mathcal{N}} \left[n_i \ln\left(\frac{g_i}{n_i} - a\right) - \frac{g_i}{a} \ln\left(1 - a\frac{n_i}{g_i}\right)\right] \tag{13}$$

where g_i is the degeneracy, and $a = 0, +1, -1$ for classical particles, fermions and bosons, respectively.

The purpose of this paper is to generalize some previous works [4–7] by means of information theory and derive a more general form for the Eddington factor. We will study the case of a radiation gas in the low occupation number approximation (in order to obtain analytical expressions) submitted both to an energy flux and a particle flux. The reason is that, in order to obtain a nonpurely advective energy flux, as in the Lorentz case and to generalize the study performed by Minerbo, we must consider both the constraints of fixed energy density e, energy flux $\boldsymbol{J}_E$ and particle flow $\boldsymbol{J}_N$. If $\boldsymbol{J}_E$ and $\boldsymbol{J}_N$ are taken as independent variables, it is possible to demand that the particle flow is null in order to eliminate any advective contribution from the energy flux so that it reduces to a pure heat flux. Notice that the distribution function that maximizes the entropy can no longer be an equilibrium one, as there is no reference frame in which one could find an equilibrium system simultaneously at rest (i.e. with no photon flow) and with an energy flux. In addition, from the general expressions obtained in this section, one can recover in the appropriate limit, expressions previously obtained in the literature for radiation submitted to an energy flux and particle flux.

Let us note that several reasons can induce a non-Planckian distribution for a photon gas coupled to matter in a nonequilibrium state submitted to high gradients or rapidly varying fluxes. On the one hand, photons can interact weakly with matter in the timescale over which the flow variables change. For this reason, the temperature of the photon gas can be different from the local equilibrium temperature of the matter. In these situations, a Stefan–Boltzmann-like law, namely aT_R^4, may be used with a temperature T_R different from the local equilibrium temperature T_M of the matter. This situation appears, for example in the so-called diluted radiation [18], of interest in photo voltaic devices. On the other hand, if we consider matter submitted to large temperature gradients and the length scale of its interaction with photons is large in comparison with the scale of variation of temperature in matter, the distribution of photons can be anisotropic. The previous examples are attempts to include these effects in the statistical distribution of photons and in the Eddington factor.

The distribution function to consider will then be

$$n(\boldsymbol{p}) = \frac{2}{[\exp(\beta pc + \boldsymbol{I}\cdot pc\boldsymbol{c} + \boldsymbol{K}\cdot\boldsymbol{c}) - 1]} \approx 2\ \exp[-\beta pc - \boldsymbol{I}\cdot pc\boldsymbol{c} - \boldsymbol{K}\cdot\boldsymbol{c}] \tag{14}$$

where $\boldsymbol{I}$, $\boldsymbol{K}$ are the Lagrange multipliers related to energy flux and particle flux respectively and the factor of 2 is related to the two possible polarizations of photons. If $\boldsymbol{I} = 0$ the constrained flux is the particle flux and one recovers the results obtained by Minerbo [5], while when one takes $\boldsymbol{K} = 0$, the constrained flux is the energy flux and the Lorentz limit is recovered. Any integral quantity defined in terms of this latter-approximated distribution converges provided that $\beta > |\boldsymbol{I}|c$. In addition, in order to simplify the calculations, we will assume that $\boldsymbol{I}$ and $\boldsymbol{K}$ are parallel. With these requirements we can find any thermodynamic quantity in terms of the special functions defined by:

$$\Psi_n(a,b) \equiv \int_{-1}^{1} \frac{\mathrm{e}^{-ax}}{(1+bx)^n}\,\mathrm{d}x \tag{15}$$

which verify the useful properties:

$$\frac{\partial\Psi_n}{\partial a} = \frac{1}{b}[\Psi_n - \Psi_{n-1}] \tag{16}$$

$$\frac{\partial\Psi_n}{\partial b} = \frac{n}{b}[\Psi_{n+1} - \Psi_n]. \tag{17}$$

In particular, the Lagrange multipliers $\boldsymbol{K}$, $\boldsymbol{I}$ can be related with the dissipative fluxes by using (14) in equations (2) and (8), leading to new nonequilibrium equations of state:

$$J_E = \frac{24\pi c}{(hc)^3\beta^4}\frac{1}{b}[\Psi_3(a,b) - \Psi_4(a,b)] \tag{18}$$

$$J_N = \frac{8\pi c}{(\beta hc)^3}\frac{1}{b}[\Psi_2(a,b) - \Psi_3(a,b)] \tag{19}$$

where we have defined $a := |\boldsymbol{K}|c$ and $b := |\boldsymbol{I}|c/\beta$, while the caloric equation of state can be computed from (1) and is given by:

$$e = \frac{24\pi}{(hc)^3\beta^4}\Psi_4(a,b). \tag{20}$$

Note that the presence of dissipative fluxes is seen to modify the Stefan–Boltzmann law. The density of photons can also be computed from (7) and is given by:

$$\rho = \frac{8\pi}{(hc\beta)^3}\Psi_3(a,b). \tag{21}$$

The reduced fluxes f_E and f_N are given by:

$$f_E = \frac{1}{b}\left[\frac{\Psi_3(a,b)}{\Psi_4(a,b)} - 1\right] \tag{22}$$

$$f_N = \frac{1}{b}\left[\frac{\Psi_2(a,b)}{\Psi_3(a,b)} - 1\right] \tag{23}$$

so they do not coincide in general. The Eddington factor calculated from (3) and (6) is given by:

$$\chi = \frac{1}{b^2}\left[1 - 2\frac{\Psi_3(a,b)}{\Psi_4(a,b)} + \frac{\Psi_2(a,b)}{\Psi_4(a,b)}\right] = \frac{1}{b}[f_N - f_E + bf_N f_E] \tag{24}$$

and, thus, it is given in parametric form as a function of both fluxes, f_E and f_N by equations (22)–(24). The behaviour of this Eddington factor for low flux values is analysed below, but let us note that in equilibrium ($f_N = f_E = 0$), the isotropic Eddington factor $\chi = \frac{1}{3}$ is recovered. Now, we discuss equation (24) in some important particular cases.

We start by considering the situation of a pure heat flow, which corresponds to energy transport without net mass flow, i.e. we impose that $f_N = 0$. From equation (19), we can easily observe that this condition is simply given by:

$$\Psi_2(a, b) = \Psi_3(a, b) \tag{25}$$

thus a and b are no longer independent variables. Using equations (22) and (25) one can express a and b as a function of f_E. In this case of pure heat flux, the Eddington factor adopts a simpler form,

$$\chi = \frac{1}{b^2}\left[1 - \frac{\Psi_2(a, b)}{\Psi_4(a, b)}\right] = -\frac{f_E}{b}. \tag{26}$$

Here we recall that first the photon flux f_N is constrained and then settled to 0 *a posteriori*.

If the photon flux f_N is unconstrained, which corresponds to the limit $a = 0$, f_N is related to f_E by:

$$f_N = \frac{1}{f_E}\left(2 - \sqrt{4 - 3f_E^2}\right) \tag{27}$$

and one recovers the so-called Lorentz Eddington factor, namely

$$\chi = \tfrac{5}{3} - \tfrac{2}{3}\sqrt{4 - 3f_E^2}. \tag{28}$$

This Eddington factor was proven to correspond to an equilibrium moving system in [9]: the constraint of given energy flux $\boldsymbol{J}_E$ (or equivalently, given momentum $\boldsymbol{P}$, as $\boldsymbol{J}_E = c^2\boldsymbol{P}$) can be realized with a (local-)equilibrium moving system if one does not constrain the particle flow of the system.

In the limit $b \to 0$ (fixed particle flux $\boldsymbol{J}_N$ but unconstrained $\boldsymbol{J}_E$) we obtain a simpler expression for the functions $\Psi_n(a, b = 0)$

$$\Psi_n(a, b = 0) = \int_{-1}^{1} \exp(-ax)\,\mathrm{d}x = 2\frac{\sinh a}{a} \tag{29}$$

which used in equations (18)–(21) and (24) leads to the parametric Eddington factor:

$$\begin{aligned} f_E &= f_N = \frac{1}{a} - \coth a \\ \chi &= 1 - \frac{2}{a}\left(\frac{1}{a} - \coth a\right). \end{aligned} \tag{30}$$

Equations (30) were obtained by Minerbo [5] in one of the first attempts to obtain a variable Eddington factor for anisotropic nonequilibrium radiation from probabilistic arguments. In contrast to the Lorentz case, it corresponds to a true nonequilibrium situation: the distribution of photons (14) with $b = 0$ cannot be transformed into a Planckian by a Lorentz transformation.

In figure 1 we compare the Eddington factors (28), (30), (26) and an expression arising from a Chapmann–Enskog calculation and expressed in a parametric form by [2]:

$$\begin{aligned} f_E &= 1/m - \coth m \\ \chi &= -\coth m(1/m - \coth m). \end{aligned} \tag{31}$$

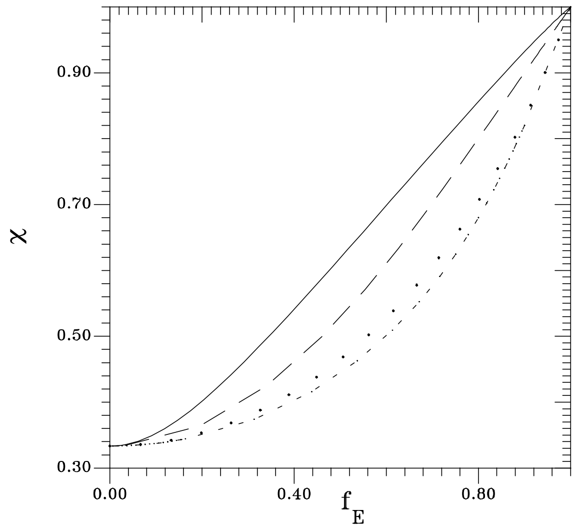

Figure 1. Functional relationship between χ and f_E for four different models. Full curve: pure heat flux; broken curve: Levermore (equations (72) and (73)); short broken curve: Minerbo ($b = 0$); dotted curve: Lorentz ($a = 0$).

Note that the pure heat flux case, equation (26) grows with the energy flux more rapidly than those of Lorentz and Minerbo as any advective contribution to the energy flux has been subtracted by demanding that $f_N = 0$.

The entropy density is also modified due to the external fluxes, yielding

$$s(e, J_E, J_N) = \frac{8\pi}{(hc\beta)^3} k_B \left[\left(4 - \frac{a}{b} \right) \Psi_3(a, b) + \frac{a}{b} \Psi_2(a, b) \right]. \tag{32}$$

In the limit of vanishing fluxes, $a \to 0$, $b \to 0$ we recover the equilibrium entropy multiplied by a factor of $90/\pi^4 \simeq 0.92$ which is due to the approximation of neglecting the -1 term in the distribution function (14).

In the case of pure heat flux, equation (32) takes a more simpler form. Using equations (18) and (25) one can express a and b as a function of J_E and we can write for the entropy

$$s(e, J_E) = \frac{32\pi}{(hc\beta)^3} k_B \Psi_3(a, b) = 4k_B\rho. \tag{33}$$

The relation coming from the last equality is also known to hold in equilibrium.

The nonequilibrium entropy density (32) in the Minerbo's limit $b = 0$, is given by

$$s(e, \boldsymbol{J}_N) = \frac{45}{2\pi^4} s_{eq}(e) \left(\frac{a}{\sinh a} \right)^{3/4} \left[5 \frac{\sinh a}{a} - \cosh a \right] \tag{34}$$

and the energy and photon densities are given by

$$\begin{aligned} e &= \frac{48\pi}{(hc)^3\beta^4} \frac{\sinh a}{a} \\ \rho &= \frac{16\pi}{(hc\beta)^3} \frac{\sinh a}{a}. \end{aligned} \tag{35}$$

Note that, the larger the flux, the lower the entropy, revealing the larger order existing in the system. In the limit $a \to \infty$, i.e. all photons moving collectively, the entropy $s \to -\infty$ and $\beta \to \infty$, while in the limit $a \to 0$ we recover the equilibrium situation.

Once the nonequilibrium entropy density is known, the thermodynamic pressure can be obtained from the relation:

$$s = \frac{e}{T} + \frac{p}{T} + k_B \boldsymbol{I} \cdot \boldsymbol{J}_E + k_B \boldsymbol{K} \cdot \boldsymbol{J}_N \tag{36}$$

(being $T = 1/k_B\beta$) and it verifies:

$$p = \frac{8\pi}{(hc)^3\beta^4}\Psi_3(a,b) = \frac{\rho}{\beta} \tag{37}$$

as in equilibrium. However, now the equilibrium relation between the thermodynamic pressure p and the trace of the pressure tensor, namely $\mathrm{Tr}(\boldsymbol{P}_E) = 3p$, is not satisfied. This can be shown by noting that the trace of the pressure tensor is $\mathrm{Tr}(\boldsymbol{P}_E) = e$ for relativistic particles, and that the relation between the thermodynamic pressure p and the energy density e as obtained from equations (20), (37),

$$p = \frac{e}{3}\frac{\Psi_3(a,b)}{\Psi_4(a,b)} = \frac{e}{3} + \frac{b}{3c}J_E \tag{38}$$

differs from the equilibrium one, $p = e/3$. From equation (38), one obtains that the relation $p = e/3$ holds both in the Minerbo's case ($b = 0$) and in equilibrium at rest ($J_E = 0$).

The thermodynamic pressure p is not related either to the isotropic part of the pressure tensor, namely $e(1-\chi)/2$, in the general situation as shown by inspection of equations (20), (24) and (37): $p = e(1-\chi)/2$ only in the Lorentz's case and in equilibrium at rest.

If we restrict ourselves up to second order in $\boldsymbol{J}_E$ and $\boldsymbol{J}_N$, we can find simpler analytical expressions for the thermodynamic quantities than the previous ones, which allows a simpler physical interpretation. First, a simple relation between the Lagrange multipliers and fluxes is seen to hold:

$$f_N = -\frac{a}{3} - b \qquad f_E = -\frac{a}{3} - \frac{4b}{3} \tag{39}$$

and we can, thus, obtain an explicit flux-dependent caloric equation of state up to second order:

$$\beta \simeq \beta_0[1 + \tfrac{3}{8}(5f_E^2 - 8\boldsymbol{f}_N \cdot \boldsymbol{f}_E + 4f_N^2)] \qquad \text{with } \beta_0 := \left[\frac{24\pi}{(hc)^3 e}\right]^{1/4}. \tag{40}$$

Note that $T_0 = 1/k_B\beta_0$ is the temperature corresponding to equilibrium radiation with the same energy density e. Thus, equation (40) shows that the nonequilibrium mean temperature defined as $T = 1/k_B\beta$ is smaller than the corresponding local equilibrium temperature T_0, as expected from the general arguments of Landau [19] which must hold for all possible nonequilibrium distributions of radiation.

The entropy density of the system can also be written as the equilibrium one plus a quadratic correction in the fluxes:

$$s(e, J_E, J_N) = \frac{64\pi}{(hc\beta_0)^3}k_B\left[1 - \frac{3}{2}\left(\frac{3}{4}f_E^2 - \frac{3}{2}\boldsymbol{f}_N \cdot \boldsymbol{f}_E + f_N^2\right)\right] \tag{41}$$

and, as expected, the flux-dependent contribution reduces the value of the entropy, in agreement with the fact that it corresponds to a more ordered physical situation.

The particle density verifies

$$\rho = \frac{16\pi}{(hc\beta_0)^3}\left[1 - \frac{3}{2}\left(\frac{3}{4}f_E^2 - f_N^2\right)\right] \tag{42}$$

and the Eddington factor can be approximated by

$$\chi \simeq \tfrac{1}{3} + \tfrac{2}{5}[5f_E^2 - 8\boldsymbol{f}_E \cdot \boldsymbol{f}_N + 4f_N^2]. \tag{43}$$

Let us observe that both fluxes must be null in order to recover the isotropic Eddington factor $\chi = \frac{1}{3}$, due to the fact that the fluxes f_N, f_E are considered as independent variables in the derivation. This situation is different from those encountered in the Eddington factor depending on two parameters introduced in [4], where f_N and f_E are considered as dependent variables and the Eddington factor depends on f_E and a nonequilibrium chemical potential for the photons (which must be set by hand to zero at equilibrium).

We should also note that the correction to the isotropic Eddington factor is always positive (the bilinear form is always positive) up to second order in the fluxes. In addition, from equation (43) we can observe that the case of pure heat flux ($f_N = 0$) corresponds, among the situations we have considered, to the case in which the correction to the isotropic value $\frac{1}{3}$ due to f_E is higher: from equation (43) it is seen to be four times higher than that observed in the Lorentz situation ($f_N = 3/4 f_E$) and five times higher than that of Minerbo ($f_N = f_E$).

3. Anisotropic radiation with chemical potential

Up until now, we have considered, as usual, that photons have a null chemical potential, so its number is undetermined. However, the use of a nonzero chemical potential for a photon gas has already been proposed in several (and physically distinct) situations. For example, in solid-state physics, nonequilibrium but steady-state quasi-Fermi distributions have been used since the 1950s for electron gas. In this case, the study of a photon gas coupled with this electron gas via adsorption–emission processes leads [18] to a Bose–Einstein distribution

$$n(\epsilon) = \frac{2}{\exp(\beta\epsilon + \mu_{ph}\beta) - 1} \tag{44}$$

where μ_{ph} is the chemical potential of the photons. Such a non-null chemical potential is related to stimulated emission of photons and other out-of-equilibrium situations. In astrophysics, massless particles such as neutrinos and photons have been considered with nonzero chemical potential in nonequilibrium situations, when the number-conserving Compton scattering dominates (see for example [4] and references therein).

From a technical point of view, the occupation number given in (14) leads to divergent-integrated quantities out of the low occupation number approximation. This drawback can also be removed by introducing a nonequilibrium chemical potential. In addition, if we consider a photon gas with energy density e which transports an energy flux $\boldsymbol{J}_E$ with a fixed particle flow $\boldsymbol{J}_N$, it seems quite reasonable to impose, together with these constraints to the entropy, the constant mean number of particles ρ, being this latter imposed in order to be consistent with the fact of imposing a fixed value for the particle flow. This assumption implies, as in the examples considered above, that photons out of equilibrium may have a non-null chemical potential. The occupation number which maximizes the entropy (13) under these constraints is:

$$n = \frac{2}{\exp(\beta pc + \mu_{ph}\beta + \boldsymbol{I}\cdot pc\boldsymbol{c} + \boldsymbol{K}\cdot\boldsymbol{c}) - 1}. \tag{45}$$

Once the distribution function is known, one can obtain the thermodynamics of the system. As above, we restrict ourselves to the low occupation number limit in order to make a comparison with the previous results. This limit corresponds to an ultrarelativistic classical gas (apart from the factor of 2 related to the polarization of photons). Following the same procedure as in the previous sections, we can easily obtain in this approximation (assuming as in the previous section, that $\beta > |\boldsymbol{I}|c$ so that integrations converge and $\boldsymbol{I}$ and $\boldsymbol{K}$ are parallel for simplicity). With these requirements we can find ($a \equiv |\boldsymbol{K}|c$ and $b \equiv |\boldsymbol{I}|c/\beta$):

$$\rho = \frac{8\pi}{(hc\beta)^3} \exp(-\mu_{ph}\beta)\Psi_3(a,b) \tag{46}$$

$$e = \frac{24\pi}{(hc\beta)^3\beta} \exp(-\mu_{ph}\beta)\Psi_4(a,b) \tag{47}$$

$$J_E = \frac{24\pi c}{(hc\beta)^3\beta} \exp(-\mu_{ph}\beta)\frac{1}{b}[\Psi_3(a,b) - \Psi_4(a,b)] \tag{48}$$

$$J_N = \frac{8\pi c}{(hc\beta)^3} \exp(-\mu_{ph}\beta)\frac{1}{b}[\Psi_2(a,b) - \Psi_3(a,b)]. \tag{49}$$

On the other hand, the entropy density can be written as

$$s = k_B\rho\left[\ln\left(\frac{8\pi}{(h\beta c)^3\rho}\right) + 4 + \ln\Psi_3(a,b) + \frac{a}{b}\left(\frac{\Psi_2(a,b)}{\Psi_3(a,b)} - 1\right)\right] \tag{50}$$

and the thermodynamic pressure is also given by equation (38).

With the exception of pressure, all these quantities differ from those introduced in the previous section. However, the reduced variables, namely f_E, f_N and χ are the same, as the new contribution due to chemical potential vanishes. Therefore, this new approach does not affect the study of radiative transfer, though it clearly modifies the thermodynamics of the system. Note, however, that the new non-null nonequilibrium chemical potential for photons can be written, in terms of the particle density of the system as

$$\mu_{ph} = \frac{1}{\beta}\ln\left(\frac{8\pi}{\rho(h\beta c)^3}\Psi_3\right) \tag{51}$$

so it does not reduce to zero for null fluxes, unless we also require independently that ρ is given by the equilibrium expression $\rho = 16\pi/(h\beta c)^3$. This fact arises directly from the hypothesis that a fixed particle density can be fixed independently from the fluxes. However, this is also the case, for instance, in the treatment performed by Fu [4].

In order to make a comparison with the results in the previous section, we can rewrite the particle density ρ as

$$\rho = \frac{8\pi}{(h\beta c)^3}\Psi_3(a,b)(1+\alpha) \tag{52}$$

where α is the modification with respect to the particle density in (21), so the rest of the previous quantities can also be found to be given by

$$e = \frac{24\pi}{(h\beta c)^3\beta}(1+\alpha)\Psi_4(a,b) \tag{53}$$

$$J_E = \frac{24\pi c}{(h\beta c)^3\beta}(1+\alpha)\frac{1}{b}(\Psi_3(a,b) - \Psi_4(a,b)) \tag{54}$$

$$J_N = \frac{8\pi c}{(h\beta c)^3}(1+\alpha)\frac{1}{b}(\Psi_2(a,b) - \Psi_3(a,b)) \tag{55}$$

so that, if the chemical potential is not null, the thermodynamic equations of state introduced in section 3 will be modified by a factor of $(1+\alpha)$.

4. Conclusions

In this paper we have applied information-theory-based nonequilibrium statistical mechanics to describe anisotropic radiation in the so-called variable Eddington factor scheme, investigating the main thermodynamic features of the system.

Typically, in the variable Eddington factor closure scheme, one considers the reduced energy flux $\boldsymbol{f}_E$ as a new variable and the pressure tensor is written as a function of it to close the hierarchy of radiative transfer equations. However, this procedure, widely used in the applications of the radiative transfer equation, has some thermodynamic consequences which are usually skipped. The main consequence is that the local-equilibrium hypothesis must be abandoned as it is not expected to provide any dependence in the energy flux for the pressure tensor except in the case in which we consider radiation in a moving frame. The local-equilibrium hypothesis implies a local black-body distribution for radiation, but it is known [10] that in many radiation transfer problems the photon distribution strongly departs from this simpler black-body expression.

The use of information theory allows one to obtain a distribution function for the photon gas consistent with this closure scheme, and the flux-dependent Eddington factor as well as the thermodynamics of the system can be obtained. In this formalism, the flux dependence of the Eddington factor merely reflects the fact that the fluxes characterizing the anisotropy of the system are incorporated as new thermodynamical variables. The necessity of incorporating dissipative fluxes within the set of thermodynamic variables beyond the local equilibrium hypothesis has been introduced by recent nonequilibrium thermodynamic theories [20–22].

In the variable Eddington closure scheme we have two fluxes in the system: the energy flux $\boldsymbol{J}_E$ and the particle flux $\boldsymbol{J}_N$. If these moments are taken as dependent variables, we must only use one of them as a constraint in the informational entropy. The effect of considering each of them as a constraint has already been performed [5–8], whereas the effect of considering them as independent variables is the subject of sections 2 and 3.

If the energy flux $\boldsymbol{J}_E$ is taken as a constraint in the maximization of the informational entropy, the results obtained by a purely macroscopic method in [6, 7] are recovered. As already pointed in [9], the results obtained in [6–8] were not really a description of nonequilibrium radiation, but merely equilibrium radiation as observed from a moving reference frame. Thus, these procedures allow the study of anisotropic radiation but, being the anisotropy due to the relative motion between matter and radiation (such as in the case of the cosmic background radiation), they do not lead to a nonequilibrium situation. Both information theory and the macroscopic approach followed in [6, 7] consider an entropy depending on the energy flux but no restrictions about the global motion of the system are imposed. Therefore, when the condition of maximum entropy is used, an equilibrium moving system appears because equilibrium situations have the maximum entropy and the moving system verifies the imposed constraint of a nonzero energy flux. In fact, irreversibility is related to positive entropy production and this fact, intrinsically related to the nature of the processes occurring in the system, cannot be changed by simply performing a Lorentz boost.

Nevertheless, from the point of view of nonequilibrium thermodynamics beyond local equilibrium, we can observe that, according to the results in [8], the existence of a flux-dependent temperature might seem a physically reasonable assumption in this context, as it arises from a purely equilibrium situation. This would suggest a further validity of flux-dependent equations of state for a more general, nonequilibrium situation.

To obtain a true nonequilibrium situation, one can impose the particle flux $\boldsymbol{J}_N$ as a constraint, instead of the energy flux $\boldsymbol{J}_E$ as done by Minerbo [5]. His definitions were frequency dependent and ours are not. However, if we consider a grey medium and the radiation intensity factorizes into a frequency- and angular-dependent part, Minerbo's Eddington factor can be recovered applying information theory (with integrated frequencies) submitted to a fixed particle flow.

In section 2, by introducing both fluxes $\boldsymbol{J}_E$, $\boldsymbol{J}_N$ as independent variables, we have obtained a unified treatment that allows the rederivation of both Lorentz's and Minerbo's cases in the proper limits. In addition, we obtain new more general forms for the (two-parameter) Eddington factor that may be useful when the matter velocity is not null with respect to the photon flow and such an effect introduces an advective contribution to the energy flux. Hence, we have considered the Eddington factor describing radiation under a pure heat flow (particle flow is set to zero), for which the anisotropic effects due to the flux seem to be higher than in any of the previous situations.

In addition to nonequilibrium Eddington factors, we have also studied the nonequilibrium equations of state arising from such a formalism. Note that, whereas the equation

$$p = \rho k_B T \tag{56}$$

always remains, the nonequilibrium temperature T is related to energy by:

$$\frac{1}{T} = k_B \left(\frac{24\pi}{(hc)^3 e} \Psi_4(a, b) \right)^{1/4} \tag{57}$$

so a nonequilibrium, flux-dependent temperature appears. This fact is in complete agreement with the predictions of some macroscopic theories [23–25] that have introduced nonequilibrium equations of state. Note also that $\Psi_4(a, b) \geqslant 1$, so the nonequilibrium temperature T is smaller than the equilibrium one. With regard to the thermodynamic pressure p, given by

$$p = \frac{e}{3} + \frac{J_E b}{3c} \tag{58}$$

it is not related in the general case neither to the trace of the pressure tensor, as usual, nor to its isotropic part. This situation is, in fact, analogous to that encountered in [8] for a classical ideal gas submitted to a heat flux. However, if $b = 0$, $p = e/3$ and it is thus related to the trace of the pressure tensor; whereas in the case of an equilibrium moving system ($a = 0$), $p = e\frac{1-\chi}{2}$, i.e. it is given by the isotropic part of the pressure tensor.

Finally, we have also considered the case in which a nonequilibrium chemical potential for photons is introduced as suggested in [18] and done in [4]. Although the necessity of such an assumption is not clear, it seems a plausible ansatz which also removes the technical difficulties of convergence beyond the low occupation number limit. Within this limit, if μ_{ph} is introduced, the radiative properties of the system are not modified, but in the thermodynamic equations of state a new factor $(1 + \alpha)$ appears. The general case deserves further study and will be the object of a future paper.

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